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New Foundations for Rewriting Logic

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Abstract

This paper presents an abstract treatment of the foundations of rewriting logic, generalising in three ways: an arbitrary 2-category plays the role of the specific 2-category $\mathbb{C}$at; the foundations are rendered fully independent of the underlying equational logic; and the semantics covers non-conventional sentences. Despite the high level of abstraction a number of properties of interest to Computing Science are seen to hold including the existence of initial models, liberality along theory morphisms, exactness, and a completeness theorem for a very abstract notion of deduction.

1 Introduction

This paper sets out to cast new light upon the foundations of *Rewriting Logic* (henceforth RWL) [11] by formulating them suitably abstractly. This abstract formulation makes sense in any 2-category (not just $\mathbb{C}$at), is fully independent of the underlying equational logic, and provides semantics for sentences which go beyond those of conventional rewriting logic. Thus it paves the way for the development of non-conventional rewriting logics over non-conventional equational logics. From that perspective the aforementioned new light is similar to that cast upon the subject of algebraic specification by the *Theory of Institutions* [7], or upon equational logic by the *Category-based Equational Logic* (henceforth CBEL) of [2].

The theory presented here – called *Universal Syntax* (henceforth USX) – is developed in detail in [8], and the reader is referred to it for most of the proofs. The mathematics underlying it all belongs to a branch of category theory called "enriched category theory" – the basic idea of which is that morphisms between two objects a and b form an *object* in a fixed *base* category (sometimes called *universe*) $\mathbb{V}$, rather than a mere set (an object in $\mathbb{S}$et). For suitable $\mathbb{V}$ most of ordinary category theory generalises to enriched category theory. The reader unfamiliar with the latter will find it helpful to think of $\mathbb{V}$ as being $\mathbb{S}\mathrm{et}$ or $\mathbb{C}\mathrm{at}$ – the universe pertinent to RWL – when reading the sections devoted to the more general theory. The following paragraphs are devoted to an account of the philosophy and motivation underlying USX, beginning with signatures, perhaps the most basic of its features, and illustrated with notions from RWL.

A USX signature is an arbitrary "models-to-domains" $\mathbb{V}$-functor $\xi : M \to X$, a choice which was inspired by the success of that concept (with $\mathbb{V} = \mathbb{S}\mathrm{et}$) in the theory of CBEL. Such a conceptualisation of signature is based on the fact that any RWL S-sorted signature Σ gives rise to a forgetful 2-functor (i.e., $\mathbb{V} = \mathbb{C}\mathrm{at}$) $U_\Sigma : \mathrm{Mod}(\Sigma) \to [S, \mathbb{C}\mathrm{at}]$, mapping Σ-models to underlying S-sorted carriers (categories), Σ-model morphisms to their underlying S-sorted functors, and Σ-modifications to their underlying S-sorted natural transformations. Furthermore an RWL signature morphism $(f, \sigma) : (S, \Sigma) \to (S', \Sigma')$ gives rise to a *reduct* 2-functor $\mathrm{Mod}(f, \sigma) : \mathrm{Mod}(S', \Sigma') \to \mathrm{Mod}(S, \Sigma)$ with the property that for any Σ'-model m, $U_\Sigma(m^\sigma) = f; U_{\Sigma'}(m)$, i.e., the underlying S-sorted carrier of the reduct of a Σ'-model is the same as the reduct of the underlying S'-sorted carrier of that model. Following CBEL, this translates to the abstract USX notion of signature morphism as a pair $(f, \sigma) : (X, \xi) \to (Y, \zeta)$ where $\sigma : N \to M$, $f : Y \to X$, $\sigma; \xi = \zeta; f$ and in addition to which f is assumed right $\mathbb{V}$-adjoint – a property of importance to term translations along signature morphisms, and in recognition of the fact that $[f, 1] : [S', \mathbb{C}\mathrm{at}] \to [S, \mathbb{C}\mathrm{at}]$ is right 2-adjoint.

Given an S-sorted category (usually discrete in practice) of variables x, the category of Σ-terms over x is the *free* Σ-model over x, which means that given a Σ-model m and an S-sorted functor $f : x \to U_\Sigma(m)$ there exists a unique Σ-model morphism $f^\flat : T_\Sigma(x) \to m$ such that $x\eta; U_\Sigma(f^\flat) = f$, where $x\eta : x \to T_\Sigma(x)$ is the inclusion of the variables in $T_\Sigma(x)$. It is this property – U_Σ is right adjoint – that furnishes Σ-terms with interpretations in an arbitrary Σ-model m: given a term t of sort s containing variables x, and given a valuation (assignment of values to variables) $f : x \to U_\Sigma(m)$, the interpretation of t in m is $U_\Sigma(f^\flat)_s(t)$, where $U_\Sigma(f^\flat)_s$ is the s-component of $U_\Sigma(f^\flat)$.

The right adjointness of signatures $\xi : M \to X$ is *axiomatic* in CBEL, facilitating notions such as equational satisfaction (two terms [1] are equal in a model if their respective interpretations in that model are equal for all valuations) without reference to any *specific* equational logic. It is a remarkable fact that a comprehensive theory of equational logic, covering both denotational and operational semantics, is possible at such a level of abstraction (see [2] for details).

For the USX approach to terms, consider again the RWL example. Given

[1] [2] also considers the semantics of *built-in* terms which are taken to lie in a fixed but arbitrary model; no right adjointness is needed for that.

a term t of sort s and a Σ-model morphism $h:\ m \to n$, the square

$$\begin{tikzcd} \llbracket x, U_\Sigma(m) \rrbracket \arrow[r, "t_m"] \arrow[d, "\llbracket 1, U_\Sigma(h) \rrbracket"'] & m_s \arrow[d, "h_s"] \\ \llbracket x, U_\Sigma(n) \rrbracket \arrow[r, "t_n"'] & n_s \end{tikzcd}$$

commutes in $\mathbb{C}\mathrm{at}$, where t_m is the interpretation of t in m and $\llbracket x, U_\Sigma(m) \rrbracket$ denotes the *category* of valuations of x in $U_\Sigma(m)$. This *naturality* property is equivalent to saying that the interpretation of Σ-terms of sort s is given by a *wedge* $\iota:\ T_\Sigma(x)_s \to \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(\_) \rrbracket, \_{s})$, i.e., a family of arrows $\iota_m:\ T_\Sigma(x)_s \to \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(m) \rrbracket, m_s)$ such that the square

$$\begin{tikzcd} T_\Sigma(x)_s \arrow[r, "\iota_n"] \arrow[d, "\iota_m"'] & \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(n) \rrbracket, n_s) \arrow[d, "\mathsf{C}\mathrm{at}(\llbracket 1, U_\Sigma(h) \rrbracket, 1)"] \\ \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(m) \rrbracket, m_s) \arrow[r, "\mathsf{C}\mathrm{at}(1, h_s)"'] & \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(m) \rrbracket, n_s) \end{tikzcd}$$

commutes in $\mathbb{C}\mathrm{at}$ for every Σ-model morphism $h:\ m \to n$, where ι_m is the mapping $t \mapsto t_m$. Furthermore the interpretation of Σ-terms of sort s in Σ-models is *universal* among all such wedges: given any other category of "names" N with an interpreting wedge $\theta:\ N \to \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(\_) \rrbracket, \_{s})$ there exists a *unique* $h:\ N \to T_\Sigma(x)_s$ such that $\theta_m = h; \iota_m$ for each Σ-model m. In other words $T_\Sigma(x)_s$ contains *all* possible "natural" names, without duplicates, over variables x. The universal pair $(T_\Sigma(x)_s, \iota)$ is called the *end* of the bifunctor $\mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(\_) \rrbracket, \_{s})$, although by the customary abuse of notation explicit mention of ι is usually avoided.

This insight is captured by the single axiom on which the theory of USX is founded:

$$\boxed{T_\xi(x) = \int_m \left[X(x, m\xi), m\xi\right]}$$

which asserts that the ξ-terms over variables x, where $\xi:\ M \to X$ is a USX signature, are obtained as the $\mathbb{V}$-end of the $\mathbb{V}$-bifunctor whose value at a pair of models (m, n) is $[X(x, m\xi), n\xi]$ – the $\mathbb{V}$-*cotensor* of $X(x, m\xi)$ and $n\xi$. It relates to the RWL example as follows: $\mathbb{V} \sim \mathbb{C}\mathrm{at}, M \sim \mathrm{Mod}(\Sigma), X \sim [S, \mathbb{C}\mathrm{at}], \xi \sim U_\Sigma, X(x, m\xi) \sim \llbracket x, U_\Sigma(m) \rrbracket$, and the cotensor $[X(x, m\xi), n\xi]$ corresponds to the 2-functor $S \to \mathbb{C}\mathrm{at}$ given by the mapping $s \mapsto \mathbb{C}\mathrm{at}(\llbracket x, U_\Sigma(m) \rrbracket, n_s)$. Agreement with the CBEL notion of (non-built-in) terms follows readily upon putting $\mathbb{V} = \mathbb{S}\mathrm{et}$ and letting ξ be right adjoint. The greater generality is in keeping with the original goal of a uniform treatment of terms and their interpretations in models in general, including those of higher order institutions where there is no hope of terms lifting to a model, let alone a free one.

Nevertheless the universality of $T_\xi(x)$ is a rich property with pleasing consequences such as the fact that $T_\xi(x)$ is a $\mathbb{V}$-monad in x – a construction well-known in enriched category theory called the *codensity monad* [5] – whose associated *Kleisli* $\mathbb{V}$-category is the source of USX substitutions (i.e., continuing the RWL example, functors of the form $v : \ y \to T_\Sigma(x)$). It is also this universality that assures the existence of canonical terms translations along signature morphisms (in RWL terms, the S-sorted functors $T_{f,\sigma}(x)$ induced by signature morphisms $(f,\sigma) : \ (S,\Sigma) \to (S',\Sigma')$, whose components are of the form $T_\Sigma(x)_s \to T_{\Sigma'}(x^f)_{f(s)}$ where x^f is the translation of the variables x as an S-sorted category to an S'-sorted category).

The architecture of USX sentences reflects the objective to capture in a single notion sort constraints (declarations to restrict the domain of an operation to an "equationally defined" subset) [14], (conditional) equations, and (conditional) rewrite rules. The basic idea is best illustrated by considering the semantics of a conditional equation; that of the conditional rewrite rule is more complicated, and may be found in Section 6. Let e be the conditional equation $\forall x\, t = t' \,\mathrm{if}\, t_1 = t'_1 \& \cdots \& t_k = t'_k$, with $t, t' \in T_\Sigma(x)_s$ and $t_i, t'_i \in T_\Sigma(x)_{s_i}$ for $i = 1, \cdots, k$. For a Σ-model m to satisfy e means, given any valuation $v : \ x \to U_\Sigma(m)$ such that $t_{i\,m}(v) = t'_{i\,m}(v)$ for $i = 1, \cdots, k$ (where $t_{i\,m}(v)$ is the interpretation of t_i in m under v), it is the case that $t_m(v) = t'_m(v)$. Letting Q denote the set of *solutions* in m of the consequent $t = t'$ – i.e., those $v : \ x \to U_\Sigma(m)$ such that $t_m(v) = t'_m(v)$ – and likewise letting P denote the solution set in m of the antecedent $t_1 = t'_1 \& \cdots \& t_k = t'_k$, the satisfaction of e in m is equivalent to the existence of a function $f : \ P \to Q$ such that the triangle

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr, "p_m"'] & & Q \arrow[dl, "q_m"] \\ & [\![x, U_\Sigma(m)]\!] & \end{tikzcd}$$

commutes in $\mathbb{S}\mathrm{et}$, where p_m and q_m are subset inclusions, and $[\![x, U_\Sigma(m)]\!]$ is the set of S-sorted functions from x to $U_\Sigma(m)$ (i.e., valuations). Note that the domain and codomain of f – P and Q – are both *limits*; so

$$\begin{tikzcd} Q \arrow[r, "q_m"] & [\![x, U_\Sigma(m)]\!] \arrow[r, shift left, "t_m"] \arrow[r, shift right, "t'_m"'] & m_s \end{tikzcd}$$

is an *equaliser* diagram. Also note that the diagram (t_m, t'_m) is determined by the diagram of substitutions

$$\begin{tikzcd} S(s, \_) \arrow[r, shift left, "\bar{t}"] \arrow[r, shift right, "\overline{t'}"'] & T_\Sigma(x) \end{tikzcd}$$

where $\bar{t}_s : \ * \mapsto t$, $\overline{t'}_s : \ * \mapsto t'$, and $\bar{t}_{s'} = \overline{t'}_{s'} =$ the empty function for $s' \neq s$.

Generalising, a USX ξ*-operation* is an arrow $\kappa : \ (\omega_0, \gamma_0) \to (\omega_1, \gamma_1)$, where the γ_i are diagrams of substitutions ($\mathbb{V}$-functors into the ξ-Kleisli $\mathbb{V}$-category), and the ω_i are diagrams, of the same shape, in $\mathbb{V}$, called *weights*. The reason for the weights is that the enriched equivalent of limit is *weighted* $\mathbb{V}$*-limit*, a non-trivial example of which is provided by the conditional rewrite rule equivalent of f (see Section 6). An *interpretation* of an operation $\kappa : (\omega_0, \gamma_0) \to (\omega_1, \gamma_1)$ in a ξ-model m is an arrow $\kappa_m : \{\omega_0, \gamma_0(m)\} \to \{\omega_1, \gamma_1(m)\}$ in $\mathbb{V}$; the $\gamma_i(m)$ are the diagrams in $\mathbb{V}$ determined by the corresponding diagrams of substitutions (the γ_i) and the model m, and the $\{\omega_i, \gamma_i(m)\}$ are their $\mathbb{V}$-limits weighted by their respective weights. These operations can be organised into signatures – sets sorted by domain/codomain pairs – Π, related to but *different* from the signatures ξ of the underlying equational logic. A (ξ, Π)-model is then a pair (m, κ_m).

A USX (ξ, Π)-*sentence* is a commutative triangle involving a ξ-operation and a pair of limit *projections*, such as the triangle above, and comes in two flavours: *structural* – where the operation lies in Π – and *existential* – where it does not. The difference is in their satisfaction in a (ξ, Π)-model m: the former means the diagram involving κ_m commutes; the latter means *there exists an* f such that the diagram commutes. The rewrite rules in an RWL specification are structural because their denotation in an R-system – a rewrite model [11] – is considered *part* of the system. On the other hand RWL *sequents* are existential because they are satisfied by an R-system if an appropriate "f" exists. Finally in the case of the conditional equation above it doesn't matter as both amount to the same thing: there can only ever be one f making that triangle commute.

The paper is structured as follows. Section 2 contains a *very brief* sketch of the prerequisite enriched category theory; there is simply no space for more. As mentioned earlier the material in this paper makes perfect sense for $\mathbb{V} = \mathbb{C}\mathrm{at}$ or $\mathbb{S}\mathrm{et}$, although the section on rewriting logic does require $\mathbb{V} = \mathbb{C}\mathrm{at}$. See [1] for a gentle introduction to the subject, and [9] for a thorough treatment; neither treats the theory of $\mathbb{V}$-monads, for which the reader is referred to [5] and the very elegant [13]. The basic definitions and facts about universal syntax are covered in Section 3: signatorials, terms, terms translations, substitutions, and the special case when the signatorials involved are right $\mathbb{V}$-adjoint. The theory of institutions is reviewed very briefly in Section 4, along with some generalisations to the general $\mathbb{V}$-case, whilst Section 5 introduces USX operations and signatures, models, sentences, and satisfaction, and shows they form an institution; it goes on to discuss liberality, initial models, deduction and exactness. In Section 6 the general results of the previous sections are applied to characterising rewriting logic in a 2-category.

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2 Preliminaries

2.0.1 Symmetric Monoidal Closed Categories

A *monoidal* category $\mathbb{V}$ consists of: a category $\mathbb{V}$; a bifunctor $\square : \mathbb{V} \times \mathbb{V} \to \mathbb{V}$ called the *tensor product*; an object $e \in \mathbb{V}$ called the *unit*; isomorphisms $\alpha : (a \square b) \square c \cong a \square (b \square c)$, $\lambda : e \square a \cong a$, and $\varrho : a \square e \cong a$, natural, respectively, in a, b, c, and a, and a; all subject to certain diagrams called *coherence diagrams.*

$\mathbb{V}$ is *symmetric monoidal closed* if in addition there are isomorphisms $\gamma : a \square b \cong b \square a$ natural in a and b; and if the functor $\_ \square b : \mathbb{V} \to \mathbb{V}$ has a right adjoint denoted $[b, \_] : \mathbb{V} \to \mathbb{V}$. When two arrows f and g correspond to each other under the isomorphism $\mathbb{V}(a \square b, c) \cong \mathbb{V}(a, [b, c])$ they are said to *correspond by adjunction.*

The category $\mathbb{S}\mathrm{et}$ of sets and functions is symmetric monoidal closed, with $e = 1$ – the one point set – $\square = \times$ – cartesian product of sets – and with $[b, \_]$ given by $[b, c]$ = set of functions from b to c being right adjoint to $\times$ because of the bijection $\mathbb{S}\mathrm{et}(a \times b, c) = \mathbb{S}\mathrm{et}(a, [b, c])$. Similarly $\mathbb{C}\mathrm{at}$ is symmetric monoidal closed, with $e = 1$ – the category containing one object and no non-trivial arrows.

2.0.2 $\mathbb{V}$-Categories, $\mathbb{V}$-Functors, $\mathbb{V}$-Natural Transformations

A $\mathbb{V}$-category A consists of a set of objects $|A|$; a *hom object* $A(a, b)$ in $\mathbb{V}$ whenever $a, b \in |A|$; a *composition* $c : A(a, b) \square A(b, c) \to A(a, c)$ for all a, b, c; an *identity* morphism $u_a : e \to A(a, a)$ for all a, together with associativity (of the composition) diagrams and identity (u with respect to the composition) diagrams. Given a $\mathbb{V}$-category A its *dual* A^{op} is obtained by setting $A^{\mathrm{op}}(a, b) = A(b, a)$ (and modifying the composition suitably).

A $\mathbb{V}$-*functor* $F : A \to B$ consists of a function $|F| : |A| \to |B|$; for every pair a, b, a *structural morphism* $F : A(a, b) \to B(aF, bF)$ in $\mathbb{V}$, subject to diagrams expressing the functoriality of F.

Given $F, G : A \to B$, a $\mathbb{V}$-*natural transformation* consists of a family of morphisms $\alpha_a : e \to B(aF, aG)$ satisfying a diagam expressing its naturality.

$\mathbb{V}$ itself can be provided with the structure of a $\mathbb{V}$-category. In particular the required hom objects are given by the objects $[u, v]$. When $\mathbb{V}$ is regarded as a $\mathbb{V}$-category, it is denoted V.

The $\mathbb{V}$-categories, $\mathbb{V}$-functors and $\mathbb{V}$-natural transformations form a 2-category $\mathbb{V}\mathbb{C}\mathrm{at}$; furthermore it itself can be provided with the structure of a symmetric monoidal closed category, with tensor product $\otimes$. Given $\mathbb{V}$-categories A and B, $[A, B]$ is the $\mathbb{V}$-category of $\mathbb{V}$-functors and $\mathbb{V}$-natural transformations between A and B; its hom objects are denoted $[\![F, G]\!]$ (the object, in $\mathbb{V}$, of all $\mathbb{V}$-natural transformations between F and G).

If A is a $\mathbb{V}$-category any object a in A determines hom $\mathbb{V}$-functors $A(a, \_) : A \to V$ and $A(\_, a) : A^{\mathrm{op}} \to V$, which in turn determine the hom $\mathbb{V}$-bifunctor $A(\_, \_) : A^{\mathrm{op}} \otimes A \to V$.

2.0.3 Cotensors

A $\mathbb{V}$-category A is *cotensored* if for every v in $\mathbb{V}$ and a in A there exists an object $[v, a]$ in A together with isomorphisms $A(b, [v, a]) \cong [v, A(b, a)]$ in $\mathbb{V}$ which are $\mathbb{V}$-natural in b. The object $[v, a]$ is called the *cotensor* of v and a. In fact V itself is a cotensored $\mathbb{V}$-category with $[b, [v, a]] \cong [v, [b, a]]$. If $\mathbb{V} = \mathbb{S}\mathrm{et}$ and A is an ordinary category with products then

$$A(b, \prod_{i \in I} A) \cong \prod_{i \in I} A(b, a) \cong \mathbb{S}\mathrm{et}(I, A(b, a))$$

i.e., the cotensor $[I, a]$ is just the I-th power of a.

2.0.4 Weighted Limits, Ends and Kan Extensions

Given $\mathbb{V}$-functors $F : A \to B$, $G : A \to \mathbb{V}$ the $\mathbb{V}$-*limit of* F *weighted by* G exists when:

(i) the object of $\mathbb{V}$-natural transformations $[\![G, B(b, F)]\!]$ exists for every b in B;

(ii) there exists an object L in B and isomorphisms in $\mathbb{V}$

$$\Lambda : [\![G, B(b, F)]\!] \cong B(b, L)$$

$\mathbb{V}$-natural in b.

In general the limit L is denoted $\{G, F\}$. Putting $b = L$ yields the "limit cone" $\mu : G \to B(L, F)$. Given an $a \in |A|$ and an $x : e \to aG$ denote by $p_{a,x}$ the "projection" $x; a\mu : \{G, F\} \to aF$. Note that if B is a functor category $[B, C]$ the projections $p_{a,x}$ are $\mathbb{V}$-natural transformations $\{G, F\} \dot{\to} aF$.

$F : A^{\mathrm{op}} \otimes A \to B$, the limit $\{A(\_, \_), F\}$ is called an *end*, and is written $\int_a aF$. Denote by $a\iota$ its *limiting wedge* $\int_a aF \to aF$, which equals the projection $p_{a,u}$ where $u : e \to A(a, a)$.

Given $\mathbb{V}$-functors $F : A \to B$ and $K : A \to C$ the *right Kan Extension* $\mathrm{Ran}_K F : C \to B$ of F along K is given, on objects, by the equation $(\mathrm{Ran}_K F)(c) = \{C(c, K), F\}$.

Denote by $\Delta_1 : A \to \mathbb{S}\mathrm{et}$ the functor which maps each object to the singleton set 1, and each arrow to the identity on 1. Similarly denote by

$\Delta_1 : B \to \mathbb{C}\mathrm{at}$ the 2-functor which maps each object to the singleton category 1, each 1-cell to the identity functor 1 on 1, and each 2-cell to the identity natural transformation on 1.

The limit of a functor $F : A \to B$ is $\{\Delta_1, F\}$, the $\mathbb{S}\mathrm{et}$-limit of F weighted by Δ_1. The 2-limit of a 2-functor $G : C \to D$ is $\{\Delta_1, G\}$, the $\mathbb{C}\mathrm{at}$-limit of G weighted by Δ_1.

2.0.5 $\mathbb{V}$-Adjunctions

If $F : A \to B$ and $G : B \to A$ then F is $\mathbb{V}$ left adjoint to G if there exist isomorphisms $A(a, bG) \cong B(aF, b)$ $\mathbb{V}$-natural in a and b.

3 Universal Syntax

As mentioned in the introduction, the concept of *universal syntax* is central to everything in this paper. It provides a uniform construction of terms and their denotations, given just a $\mathbb{V}$-category of models and a $\mathbb{V}$-functor mapping models to underlying domains. This construction has a certain universal property in the category of domains which facilitates the further construction of features of interest to the logic of computing science. These include a $\mathbb{V}$-monad of terms giving rise to $\mathbb{V}$-categories of substitutions (the $\mathbb{V}$-Kleisli categories determined by the monad); canonical term translations along reduct functors giving rise to translations of substitutions; and, further down the road, an institution of great generality yet endowed with properties desirable from a computing science point of view.

3.1 Signatorials

In [2] Diaconescu makes a convincing case for the assertion that much of what is relevant to equational logic in general depends on the "forgetful" functors (from models to underlying domains) determined by the signatures, rather than on the details of the signatures themselves. His CBEL signatures are functors $\xi : M \to X$ with varying properties depending on the intended application. So for example the completeness of equational deduction requires ξ to be faithful, whilst for the construction of ξ-terms ξ is presumed right-adjoint. The definition of *signatorial*, introduced below, generalises right-adjoint CBEL signatures.

Definition 3.1 $\mathbb{V}\mathfrak{Sig}$ *is defined to be the following category:*

- *Its objects are $\mathbb{V}$-functors $\xi : M \to X$ whose right $\mathbb{V}$-Kan Extension T_ξ along itself exists; they are often written (M, ξ, X).*
- *Its arrows are of the form $(f, \sigma) : (M, \xi, X) \to (N, \zeta, Y)$ where $\sigma : N \to M$ and $f : Y \to X$ are $\mathbb{V}$-functors, f has a chosen left $\mathbb{V}$-adjoint $f^\$$ such that $1^\$ = 1$, and satisfying the equation $\zeta; f = \sigma; \xi$.*

Composition is given by the equation $(f, \sigma); (g, \pi) = (g; f, \pi; \sigma)$. The objects *are called $\mathbb{V}$-signatorials, and the arrows are called $\mathbb{V}$-signatorial morphisms. Given a signatorial $\xi : M \to X$, M is called the models category and its objects are referred to as ξ-models; X is called the domains category and its objects are referred to as ξ-domains.*

Terminology 3.2 *The word "signatorial" was chosen to avoid confusion with the notion of* USX *signature of Section 5; and to bring out the functorial nature of the concept.*

3.2 Denotations

The defining property of T_ξ, introduced below, states that in a certain sense term substitutions are wholly determined by the totality of their denotations in models.

Definition 3.3 *Given a signatorial $\xi : M \to X$, denote by $\spadesuit$ the $\mathbb{V}$-isomorphism associated with T_ξ:*

$$X(x, yT_\xi) \cong [\![X(y, \xi), X(x, \xi)]\!]$$

where $X(y, \xi)$ is the composite $\xi; X(y, \_)$.

Notation 3.4 *Given $v : x \to yT_\xi$ and $m \in |M|$, we will usually just write mv or v_m instead of $\spadesuit(v)_m$ for the arrow $X(y, m\xi) \to X(x, m\xi)$. Note that intuitively this makes sense: a valuation of the variables y in m determines a valuation of the variables x in m by composing the substitution with the original valuation.*

A very special substitution is $x\eta_\xi : x \to xT_\xi$, the "inclusion" of the variables in x into $T_\xi(x)$ as terms. It is defined as the unique substitution whose denotation in every model m is the identity $X(x, m\xi) \to X(x, m\xi)$.

When X is cotensored Definition 3.3 has an equivalent formulation in terms of *ends*:

$$T_\xi(x) = \int_m [X(x, m\xi), m\xi]$$

The limiting wedge $m\iota : T_\xi(x) \to [X(x, m\xi), m\xi]$ may be thought of as mapping terms containing variables x to their interpretations in a model m: values in m indexed by valuations of the variables x in m.

Of great importance to the viability of the notion of arrows $x \to yT_\xi$ as substitutions is the statement of Proposition 3.5 below. The *codensity* $\mathbb{V}$-*monad* construction is well-known in enriched category theory (see [5] for example).

Proposition 3.5 *The construction of Definition 3.3 determines a $\mathbb{V}$-monad $(T_\xi, \eta_\xi, \mu_\xi)$ called the codensity monad.*

Term translations are induced by reduct functors, in the sense of Proposition 3.6, which is basically the reason why the institutional satisfaction condition (see Section 4) is true for equational logics.

Proposition 3.6 *A signatorial morphism* $(f,\sigma):\ (M,\xi,X)\to(N,\zeta,Y)$ *induces a monad opfunctor* $T_{f,\sigma}:\ (X,T_\xi)\to(Y,T_\zeta)$ *such that the diagram*

$$\begin{tikzcd}
X(y,xT_\xi) \arrow[r, "\spadesuit"] \arrow[d, "f^\$"'] & \llbracket X(x,\xi),X(y,\xi)\rrbracket \arrow[d, "{[\sigma,1]}"] \\
Y(yf^\$,xT_\xi f^\$) \arrow[d, "{Y(1,xT_{f,\sigma})}"'] & \llbracket X(x,\sigma\xi),X(y,\sigma\xi)\rrbracket \arrow[d, "{\llbracket\sharp,\flat\rrbracket}"] \\
Y(yf^\$,xf^\$T_\zeta) \arrow[r, "\spadesuit"'] & \llbracket Y(xf^\$,\zeta),Y(yf^\$,\zeta)\rrbracket
\end{tikzcd}$$

commutes.

Proposition 3.7 *The mapping* $(f,\sigma)\mapsto T_{f,\sigma}$ *is functorial.*

3.3 Substitutions

Definition 3.8 is motivated by the fact that any monad supports a notion of substitution in which substitutions may be composed and are generally well-behaved.

Definition 3.8 *Given a signatorial* $\xi:\ M\to X$ *define* $\mathcal{K}(\xi)$ *to be the Kleisli* $\mathbb{V}$*-category determined by the* $\mathbb{V}$*-monad* T_ξ.

Its objects are objects x in X; its hom objects are given by the equation $\mathcal{K}(\xi)(x,y)=X(x,yT_\xi)$.

Proposition 3.9 formalizes the mapping of substitutions to their interpretations as the act of a $\mathbb{V}$-functor. Signatorials give rise to substitution systems, and signatorial morphisms to translations between these substitution systems (Proposition 3.10); and the diagram of Proposition 3.6 can be expressed as a $\mathbb{V}$-natural isomorphism (Corollary 3.11).

Proposition 3.9 *The isomorphism* $\spadesuit$ *of Definition 3.3 determines a* $\mathbb{V}$*-functor* $\spadesuit:\ \mathcal{K}(\xi)^{\mathrm{op}}\to[M,V]$ *whose value at an object* x *is* $X(x,\xi)$*, and whose structural morphisms are the isomorphisms* $\spadesuit$ *of Definition 3.3. Denote by* $\spadesuit_m$ *the* m*-component* $\mathcal{K}(\xi)^{\mathrm{op}}\to V$ *of* $\spadesuit$*. Also let* $\clubsuit:\ M\to[\mathcal{K}(\xi)^{\mathrm{op}},V]$ *correspond to* $\spadesuit$ *by adjunction.*

Proposition 3.10 *There is a functor* $\mathcal{K}:\ \mathbb{V}\mathfrak{Sig}\to\mathbb{V}\mathbb{C}\mathrm{at}$ *whose value at a signatorial* ξ *is the* $\mathcal{K}(\xi)$ *of Definition 3.8, and whose value at a signatorial morphism* $(f,\sigma):\ (M,\xi,X)\to(N,\zeta,Y)$ *is the composite*

$$X(x,yT_\xi)\xrightarrow{f^\$}Y(xf^\$,yT_\xi f^\$)\xrightarrow{Y(1,yT_{f,\sigma})}Y(xf^\$,yf^\$T_\zeta)$$

of Proposition 3.6.

Corollary 3.11 *Given a signatorial morphism* $(f,\sigma):\ (M,\xi,X)\to(N,\zeta,Y)$, $\sharp:\ \mathcal{K}(f,\sigma)^{\mathrm{op}};\spadesuit\cong\spadesuit;[\sigma,1]$

Proof. Immediate from Proposition 3.6 and the definition of $\mathcal{K}(f,\sigma)^{\mathrm{op}}$. $\square$

3.4 Right $\mathbb{V}$-Adjoint Signatorials

In the context of right $\mathbb{V}$-adjoint signatorials various identifications can be made, such as that xT_ξ is the underlying domain of the free ξ-model over x. Denotations of substitutions, and term translations induced by signatorial morphisms also turn out as expected; in particular, putting $\mathbb{V} = \mathbb{S}\mathrm{et}$, they agree with [2].

Proposition 3.12 *If $\xi : M \to X$ has a left $\mathbb{V}$-adjoint $\xi^\$$ the following identifications may be made:*

(i) $T_\xi = \xi^\$\xi$.

(ii) *The commutative diagram*

$$\begin{tikzcd} X(x, yT_\xi) \arrow[rr, "\spadesuit_m"] \arrow[d, "X(\_,m\xi)"] & & \llbracket X(y,m\xi), X(x,m\xi)\rrbracket \\ \llbracket X(yT_\xi, m\xi), X(x,m\xi)\rrbracket \arrow[rr, "\llbracket X(1,m\varepsilon\xi),1\rrbracket"'] & & \llbracket X(yT_\xi, m\xi T_\xi), X(x,m\xi)\rrbracket \arrow[u, "\llbracket \xi^\$\xi,1\rrbracket"'] \end{tikzcd}$$

In particular this means that for $\mathbb{V} = \mathbb{S}\mathrm{et}$, $\spadesuit_m$ is the mapping

$$(v :\ x \to yT_\xi \ \longmapsto\ (w :\ y \to m\xi \ \longmapsto\ v;\overline{w} :\ x \to m\xi))$$

as expected.

(iii) $T^\sharp_{f,\sigma} = \eta_f T_\xi; xf^\$\gamma\xi$ *where $\gamma :\ f\xi^\$ \to \zeta^\$\sigma$ is uniquely defined by the equation $\eta_\zeta f = f\eta_\xi;\gamma\xi$. The expression on the right first appeared in [2].*

4 Institutions

The theory of institutions [7] is based on an abstract formulation of logical system, allowing the analysis of properties of importance to logical systems without reference to any specific logical system.

Definition 4.1 *An institution comprises*

- *a category* Sign*, whose objects and arrows are respectively called signatures and signature morphisms;*
- *a functor* $\mathrm{Mod} :\ \mathrm{Sign} \to \mathbb{C}\mathrm{at}^{\mathrm{op}}$*, mapping signatures Σ to categories of Σ-models and Σ-model morphisms, and mapping signature morphisms $\sigma :\ \Sigma \to \Sigma'$ to reduct functors* $\mathrm{Mod}(\sigma)$*;*
- *a functor* $\mathrm{Sen} :\ \mathrm{Sign} \to \mathbb{S}\mathrm{et}$*, mapping signatures Σ to sets of Σ-sentences;*
- *for each signature Σ a relation $\models_\Sigma \subseteq |\mathrm{Mod}(\Sigma)| \times \mathrm{Sen}(\Sigma)$, called Σ-satisfaction, such that for any signature morphism $\sigma :\ \Sigma \to \Sigma'$ the Satisfaction Condition*

$$m \models_{\Sigma'} \mathrm{Sen}(\sigma)(e) \textit{ iff } \mathrm{Mod}(\sigma)(m) \models_\Sigma e$$

holds for all Σ'-models m and Σ-sentences e. $\mathrm{Mod}(\sigma)(m)$ *and* $\mathrm{Sen}(\sigma)(e)$ *will usually be written, respectively,* m^σ *and* $\sigma(e)$.

For the purposes of this paper, the notion of institution is generalised to $\mathbb{V}$-*institution*, obtained by replacing $\mathbb{C}\mathrm{at}$ by $\mathbb{V}\mathbb{C}\mathrm{at}$ in Definition 4.1.

Definition 4.2 *Given a $\mathbb{V}$-institution, a specification is a pair (Σ, E) where Σ is a signature, and E is a set of Σ-sentences; it is a theory if E is closed under semantic deduction, i.e., $E \models_\Sigma e$ implies $e \in E$. A specification morphism is an arrow $\sigma : (\Sigma, E) \to (\Sigma', E')$ such that $\sigma : \Sigma \to \Sigma'$ is a signature morphism, E and E' are specifications, and $e \in E$ implies $E' \models \sigma(e)$; it is a theory morphism if E and E' are theories such that $e \in E$ implies $\sigma(e) \in E'$. Specifications and specification morphisms form a category* Spec*; theories and theory morphisms form a category denoted by* Th.

Given a $\mathbb{V}$-institution, it is a simple consequence of the satisfaction condition that $\mathrm{Mod} : \mathrm{Sign} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ extends to theories (as in the $\mathbb{V} = \mathbb{S}\mathrm{et}$ case; see [7]). Let $\mathrm{Mod}(\Sigma, E)$ be the $\mathbb{V}$-category whose objects are those Σ-models that satisfy E, and whose hom objects $\mathrm{Mod}(\Sigma, E)(m, n)$ equal $\mathrm{Mod}(\Sigma)(m, n)$. Given a theory morphism $\sigma : (\Sigma, E) \to (\Sigma', E')$ let $\mathrm{Mod}(\sigma) : \mathrm{Mod}(\Sigma', E') \to \mathrm{Mod}(\Sigma, E)$ be given by the mapping $m \mapsto m^\sigma$, with structural morphisms identical to those of $\mathrm{Mod}(\sigma : \Sigma \to \Sigma')$.

4.1 Institution Fragments

The purpose of Definition 4.3 below is to facilitate the identification of a (relatively) concrete institution as a "fragment" of a (relatively) generic institution. The fragment inherits both models and satisfaction from the generic institution.

Definition 4.3 *A $\mathbb{V}$-institution fragment of an institution* $(\mathrm{Sign}, \mathrm{Sen}, \mathrm{Mod}, \models)$ *consists of*

- *A category* Sign' *of signatures,*
- *A sentence functor* $\mathrm{Sen}' : \mathrm{Sign}' \to \mathbb{S}\mathrm{et}$,
- *A functor* $F : \mathrm{Sign}' \to \mathrm{Sign}$,
- *A natural transformation* $\alpha : \mathrm{Sen}' \to F; \mathrm{Sen}$.

Often F will be an inclusion of categories, and α an inclusion of sets (componentwise). The desired result is given by the following proposition:

Proposition 4.4 *A $\mathbb{V}$-institution fragment determines a $\mathbb{V}$-institution whose models and satisfaction are those of the given institution.*

Proof. Using the notation of Definition 4.3 define $\mathrm{Mod}' : \mathrm{Sign}' \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ to be the composite $F; \mathrm{Mod}$. Letting m be a Σ'-model and $e \in \mathrm{Sen}'(\Sigma')$ define $m \models_{\Sigma'} e$ to be $m \models_{\Sigma' F} \alpha(e)$. It remains to prove the satisfaction condition. Let $\sigma : \Sigma'_1 \to \Sigma'_2$ be a signature morphism in Sign', $e \in \mathrm{Sen}'(\Sigma'_1)$

and $m \in \mathrm{Mod}'(\Sigma_2')$. Then

$$m \models_{\Sigma_2'} \sigma(e) \text{ iff } m \models_{\Sigma_2' F} \alpha(\sigma(e)) \text{ iff } m \models_{\Sigma_2' F} F(\sigma)(\alpha(e)) \text{ iff } \sigma(m) \models_{\Sigma_1' F} \alpha(e)$$

iff $\sigma(m) \models_{\Sigma_1'} e$, where the second step follows from the naturality of α, and the third from the satisfaction condition of the given institution (noting that $F(\sigma)m = \sigma(m)$). □

4.2 Modularisation

One characteristic which it is important for a specification language to have is that it support the construction of large systems out of smaller pieces ([7], [3]). Two institutional properties relevant in this regard are *liberality* to support parameterised modules ([6], [4]), and *exactness* to support the putting together of specifications ([3]). Here both notions are adapted to the general $\mathbb{V}$ case.

Definition 4.5 *Given an institution, a theory morphism σ is liberal if the reduct $\mathbb{V}$-functor* $\mathrm{Mod}(\sigma)$ *is right $\mathbb{V}$-adjoint. The institution itself is liberal if all its theory morphisms are liberal.*

Definition 4.6 *An institution is exact if* $\mathrm{Mod} : \mathrm{Sign} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ *preserves colimits.*

Proposition 4.7 generalises the semi-exactness result of [2].

Proposition 4.7 *For any exact $\mathbb{V}$-institution,* $\mathrm{Mod} : \mathrm{Th} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ *is co-continuous.*

5 The Institution of Universal Syntax

Universal syntax and its semantics determine a vastly generic institution – in fact one for each base $\mathbb{V}$. Fragments (in the sense of Definition 4.3) of these may be identified for specific applications such as rewriting logic ($\mathbb{V} = \mathbb{C}\mathrm{at}$).

5.1 USX Signatures

Informally, a USX operation is an arrow whose domain and codomain are diagrams of ξ-substitutions, for some signatorial ξ, which, upon interpretation in a model, becomes an arrow between the limits of the interpretations (in that model) of those diagrams.

Definition 5.1 *Given signatorial morphism $\sigma : \xi \to \zeta$ and diagram of substitutions $\gamma : \Gamma \to \mathcal{K}(\xi)^{\mathrm{op}}$ denote by $\sigma(\gamma)$ the composite $\gamma; \mathcal{K}(\sigma)^{\mathrm{op}}$.*

Definition 5.2 *The category of USX signatures, denoted* $\mathrm{Sign}_{\mathsf{USX}}$, *has*

- ***as objects*** *pairs (ξ, Π) where $\xi : M \to X$ is a signatorial and Π is a set of sets indexed by domains and codomains as follows:*

$$\Pi = \langle \Pi_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)} | \omega_i : \Gamma_i \to V, \gamma_i : \Gamma_i \to \mathcal{K}(\xi)^{\mathrm{op}} \rangle$$

The elements of each $\Pi_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)}$ *are called* (ξ,Π)*-operations, or simply* Π*-operations when the* ξ *is understood from context, and are written* $\kappa:\ (\omega_0,\gamma_0)\to(\omega_1,\gamma_1)$*. The pairs* (ω_0,γ_0) *and* (ω_1,γ_1) *are called, respectively, the domain and codomain of* κ*.* ω_0 *(*ω_1*) is called the weight of the domain (codomain);* γ_0 *(*γ_1*) is called the diagram (of substitutions) of the domain (codomain).*

- ***as arrows*** *triples* $(f,\sigma,\pi):\ (\xi,\Pi)\to(\zeta,\Pi')$ *where* $(f,\sigma):\ \xi\to\zeta$ *is a signatorial morphism and* $\pi:\ \Pi\to\Pi'$ *is a set of functions indexed by* ξ *domains and codomains:*

$$\pi=\langle\pi:\ \Pi_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)}\to\Pi'_{(\omega_0,\sigma(\gamma_0)),(\omega_1,\sigma(\gamma_1))}|\omega_i:\ \Gamma_i\to V,\gamma_i:\ \Gamma_i\to\mathcal{K}(\xi)^{\mathrm{op}}\rangle$$

The composition of two signature morphisms $(f,\sigma,\pi):\ (\xi,\Pi)\to(\zeta,\Pi')$ *and* $(g,\sigma',\pi'):\ (\zeta,\Pi')\to(\theta,\Pi'')$ *is defined to be* $(fg,\sigma\sigma',\pi\pi'):\ (\xi,\Pi)\to(\theta,\Pi'')$ *where* fg *and* $\sigma\sigma'$ *are as defined in Definition 3.1, and where each component* $\pi\pi'_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)}$ *is the composite*

$$\Pi_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)}\xrightarrow{\pi}\Pi'_{(\omega_0,\sigma(\gamma_0)),(\omega_1,\sigma(\gamma_1))}\xrightarrow{\pi'}\Pi'_{(\omega_0,\sigma\sigma'(\gamma_0)),(\omega_1,\sigma\sigma'(\gamma_1))}$$

Associativity follows because the composition of signatorial morphisms is associative; and because $\mathcal{K}:\ \mathbb{V}\,\mathrm{Sign}\to\mathbb{V}\mathbb{C}\mathrm{at}$ *is a* $\mathbb{V}$*-functor (Proposition 3.10). Given* (ξ,Π)*, the morphism* $(1_X,1_M,1_\Pi)$ *is the identity on it.*

5.2 USX Models

USX-models of a signature (ξ,Π) are just what you would expect: a ξ-model m, together with interpretations in m of each of the operations in Π. Equally intuitively, (ξ,Π)-model morphisms are ξ-model morphisms which commute with the interpretations of the Π-operations.

Definition 5.3 *Given signatorial* $\xi:\ M\to X$ *and* $\mathbb{V}$*-functor* $\gamma:\ \Gamma\to\mathcal{K}(\xi)^{\mathrm{op}}$ *let* $\gamma^{\spadesuit}$ *be the composite* $\Gamma\xrightarrow{\gamma}\mathcal{K}(\xi)^{\mathrm{op}}\xrightarrow{\spadesuit}[M,V]$ *and* $\gamma^{\clubsuit}=\clubsuit;[\gamma,1]$*. Given an object* m *in* M *denote by* $\gamma^{\spadesuit}(m)$ *the composite* $\gamma;\spadesuit_m$*. In particular therefore note that* $c\gamma^{\spadesuit}(m)=X(c\gamma,m\xi)$ *for* $c\in|\Gamma|$*.*

Definition 5.4 *Given signature* (ξ,Π) *the* $\mathbb{V}$*-category of* (ξ,Π)*-models, denoted by* $\mathrm{Mod}(\xi,\Pi)$*, has*

- ***as objects*** *pairs* $(m,\langle f(\kappa):\ \{\omega_0,\gamma_0^{\spadesuit}(m)\}\to\{\omega_1,\gamma_1^{\spadesuit}(m)\}\rangle_{\kappa\in\Pi})$ *where* $m\in|M|$ *and the arrows* $f(\kappa)$ *are arbitrary. Given a* (ξ,Π)*-model* m*, denote by* m^- *its first coordinate, called the underlying* ξ*-model of* m*, and by* κ_m *the elements of its second coordinate, called the interpretation of* κ *in* m*.*

- ***as hom objects*** $\mathrm{Mod}(\xi,\Pi)(m,n)$ *– the limit in* $\mathbb{V}$ *of the diagram*

$$\begin{tikzcd} M(m^-,n^-) \arrow[r, "\gamma_1^{\clubsuit};\{\omega_1,\text{-}\}"] \arrow[d, "\gamma_0^{\clubsuit};\{\omega_0,\text{-}\}"'] & {[\{\omega_1,\gamma_1^{\spadesuit}(m^-)\},\{\omega_1,\gamma_1^{\spadesuit}(n^-)\}]} \arrow[d, "{[\kappa_m,1]}"] \\ {[\{\omega_0,\gamma_0^{\spadesuit}(m^-)\},\{\omega_0,\gamma_0^{\spadesuit}(n^-)\}]} \arrow[r, "{[1,\kappa_n]}"'] & {[\{\omega_0,\gamma_0^{\spadesuit}(m^-)\},\{\omega_1,\gamma_1^{\spadesuit}(n^-)\}]} \end{tikzcd}$$

as κ *ranges over* Π. *Denote by* U_Π *the arrow* $\mathrm{Mod}(\xi,\Pi)(m,n) \to M(m^-,n^-)$.

Proposition 5.5 *The construction of Definition 5.4 defines a* $\mathbb{V}$*-category* $\mathrm{Mod}(\xi,\Pi)$.

Proposition 5.6 *The arrows* U_Π *of Definition 5.4 are the structural morphisms of a faithful* $\mathbb{V}$*-functor* $\mathrm{Mod}(\xi,\Pi) \to M$.

Proposition 5.7 *The construction of Definition 5.4 is the object part of a functor* $\mathrm{Mod}: \ \mathrm{Sign}_{\mathsf{USX}} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$.

5.3 USX Sentences

As mentioned in the introduction a USX sentence consists of an operation, and a pair of limit projections whose targets coincide.

Definition 5.8 *A structural* (ξ,Π)*-sentence is a formal string* $\forall x\, \kappa q = p$ *where*

(i) κ *is a* (ξ,Π)*-operation*

(ii) $x = \gamma(c)$ *for some object* c *in* Γ *where*

$$\begin{tikzcd} \Gamma \arrow[r, "u_0"] \arrow[d, "u_1"'] & \Gamma_0 \arrow[d, "\gamma_0"] \\ \Gamma_1 \arrow[r, "\gamma_1"'] & \mathcal{K}(\xi)^{\mathrm{op}} \end{tikzcd}$$

is a pullback diagram in $\mathbb{V}$*, and* γ *is the diagonal* $\Gamma \to \mathcal{K}(\xi)^{\mathrm{op}}$.

(iii) p *and* q *are projections respectively of the limits* $\{\omega_0,\gamma_0^{\spadesuit}\}$ *and* $\{\omega_1,\gamma_1^{\spadesuit}\}$ *(see Definition 5.3 and Section 2.0.4); specifically they are of the form*

$$e \xrightarrow{a_i} cu_i\omega_i \xrightarrow{cu_i\mu_i} [\![\{\omega_i,\gamma_i^{\spadesuit}\}, X(x,\xi)]\!]$$

for $i = 0,1$*, where the* μ_i *are limit cones and the* a_i *are arbitrary, and noting that* $cu_i\gamma_i^{\spadesuit} = X(x,\xi)$.

An existential (ξ,Π)*-sentence is a formal string* $\exists\kappa\,\forall x\,\kappa q = p$ *subject to the conditions above except that* κ *is not presumed an element of* Π *(it's a quantified variable).*

Definition 5.9 *Define functor* $\mathrm{Sen}: \ \mathrm{Sign}_{\mathsf{USX}} \to \mathbb{S}\mathrm{et}$ *by setting* $\mathrm{Sen}(\xi,\Pi)$ *equal to the set of all* (ξ,Π) *sentences – structural and existential; and by setting* $\mathrm{Sen}(f,\sigma,\pi): \ \mathrm{Sen}(M,\xi,X,\Pi) \to \mathrm{Sen}(N,\zeta,Y,\Pi')$ *equal to the mapping*

- $\forall x\, \kappa q = p \longmapsto \forall x f^{\$}\, \pi(\kappa)\sigma(q) = \sigma(p)$
- $\exists\kappa\, \forall x\, \kappa q = p \longmapsto \exists\kappa\, \forall x f^{\$}\, \pi(\kappa)\sigma(q) = \sigma(p)$

where $\pi(\kappa)$ - the image of κ under the signature morphism $\pi: \Pi \to \Pi'$ - is as defined in Definition 5.2; and $\sigma(p)$ (and similarly $\sigma(q)$), where p is given by the composite

$$e \xrightarrow{a_0} cu_0\omega_0 \xrightarrow{cu_0\mu_0} [\![\{\omega_0, \gamma_0^{\spadesuit}\}, X(x,\xi)]\!]$$

is the composite

$$e \xrightarrow{a_0} cu_0\omega_0 \xrightarrow{cu_0\mu_0'} [\![\{\omega_0, \sigma(\gamma_0)^{\spadesuit}\}, Y(xf^{\$},\zeta)]\!]$$

where μ_0' is the limit cone of $\{\omega_0, \sigma(\gamma_0)^{\spadesuit}\}$ and $Y(xf^{\$},\zeta) = \spadesuit(xf^{\$}) = cu_0\sigma(\gamma_0)^{\spadesuit}$. Note that requirement (ii) of Definition 5.8 is also met because $cu_i = chu_i'$, where h is the canonical $\mathbb{V}$-functor $\mathrm{pullback}(\gamma_0,\gamma_1) \to \mathrm{pullback}(\sigma(\gamma_0),\sigma(\gamma_1))$, *and the u_i' are the latter's projections. Functoriality follows readily because the composition of signature morphisms is associative; because $\mathcal{K}: \mathbb{V}\,\mathrm{Sign} \to \mathbb{V}\mathbb{C}\mathrm{at}$ is a $\mathbb{V}$-functor (Proposition 3.10); and because pullbacks are universal.*

5.4 The Satisfaction Condition

Proposition 5.10 *Definitions 5.2, 5.4, and 5.9, together with Proposition 5.7 determine a $\mathbb{V}$-institution $\mathbb{V}$-USX.*

Proof. It remains to define satisfaction and to prove the Satisfaction Condition. Suppose (ξ,Π) is a USX signature and m a (ξ,Π)-model. Then $m \models_{\xi,\Pi} \forall x\, \kappa q = p$ if the triangle

$$\begin{tikzcd}
\{\omega_0, \gamma_0^{\spadesuit}(m^-)\} \arrow[rr, "\kappa_m"] \arrow[dr, "p_{m^-}"'] & & \{\omega_1, \gamma_1^{\spadesuit}(m^-)\} \arrow[dl, "q_{m^-}"] \\
& X(x, m^-\xi) &
\end{tikzcd}$$

commutes. Similarly $m \models_{\xi,\Pi} \exists\kappa\, \forall x\, \kappa q = p$ if there exists f such that $f; p_{m^-} = q_{m^-}$. Given a signature morphism $(f,\sigma,\pi): (M,\xi,X,\Pi) \to (N,\zeta,Y,\Pi')$, a (ζ,Π')-model n and a (ξ,Π)-sentence $\forall x\, \kappa q = p$, $(\sigma,\pi)(n) \models_{(\xi,\Pi)} \forall x\, \kappa q = p$ iff

$$\begin{tikzcd}
\{\omega_0, \gamma_0^{\spadesuit}(\sigma(n^-))\} \arrow[rr, "\kappa_{(\sigma,\pi)(n)}"] \arrow[dr, "p_{\sigma(n^-)}"'] \arrow[rr, phantom, shift right=6ex, "(*)"] & & \{\omega_1, \gamma_1^{\spadesuit}(\sigma(n^-))\} \arrow[dl, "q_{\sigma(n^-)}"] \\
& X(x, n^-\sigma\xi) &
\end{tikzcd}$$

commutes. This makes sense because $\sigma(n^-) = ((\sigma,\pi)(n))^-$. Now

$$\begin{tikzcd}
\{\omega_0, \gamma_0^{\spadesuit}(\sigma(n^-))\} \arrow[r, "{\{1,\flat\}}"] \arrow[d, "p_{\sigma(n^-)}"'] & \{\omega_0, \sigma(\gamma_0)^{\spadesuit}(n^-)\} \arrow[d, "\sigma(p)_{n^-}"] \\
X(x, n^-\sigma\xi) \arrow[r, "\flat"'] & Y(xf^{\$}, n^-\zeta)
\end{tikzcd}$$

commutes because $[\sigma,1]:\ [M,V] \to [N,V]$ is $\mathbb{V}$-continuous, because the isomorphism associated with a weighted limit is $\mathbb{V}$-natural, and by definition of $\{\omega_0, \_\}$; and a similar commuting square applies to $q_{\sigma(n^-)}$. Also $\kappa_{(\sigma,\pi)(n)}; \{1,\flat\} = \{1,\flat\}; \pi(\kappa)_n$ by construction of $(\sigma,\pi)(n)$ (Proposition 5.7). It follows that $(*)$ is equivalent to

$$\begin{tikzcd}
\{\omega_0, \sigma(\gamma_0)^{\spadesuit}(n^-)\} \arrow[rr, "\pi(\kappa)_n"] \arrow[dr, "\sigma(p)_{n^-}"'] & & \{\omega_1, \sigma(\gamma_1)^{\spadesuit}(n^-)\} \arrow[dl, "\sigma(q)_{n^-}"] \\
& Y(xf^{\$}, n^-\zeta) &
\end{tikzcd}$$

i.e., to $n \models_{(\zeta,\Pi')} \forall x f^{\$}\, \pi(\kappa)\sigma(q) = \sigma(p)$. A similar argument establishes the satisfaction condition for existential sentences. $\square$

5.5 Liberality

The signatorial equivalent of liberal theory morphism is

Definition 5.11 *A signatorial morphism $(f,\sigma):\ (M,\xi,X) \to (N,\zeta,Y)$ is liberal if $\sigma:\ N \to M$ has a left $\mathbb{V}$-adjoint $\sigma^{\$}$.*

Definition 5.12 *Given theory (M,ξ,X,Π,E) define $U_{\Pi,E}$ to be the composition of the canonical "$\mathbb{V}$-inclusion" $\mathrm{Mod}(\xi,\Pi,E) \to \mathrm{Mod}(\xi,\Pi)$ and U_Π, and define $U_{\xi,\Pi,E}$ to be the composite $U_{\Pi,E}\,;\xi$. Also the notation m^- for $U_\Pi(m)$ is extended to $U_{\Pi,E}(m)$ for $m \models E$.*

Theorem 5.13 below states that, under certain mild conditions usually satisfied in practice, USX theory morphisms are liberal if the underlying signatorial morphism is liberal.

Theorem 5.13 *Let $(f,\sigma,\pi):\ (M,\xi,X,\Pi,E) \to (N,\zeta,Y,\Pi',E')$ be a theory morphism such that (f,σ) is liberal, N is $\mathbb{V}$-complete, and ξ and ζ are $\mathbb{V}$-continuous. Then $\mathrm{Mod}(f,\sigma,\pi):\ \mathrm{Mod}(\zeta,\Pi',E') \to \mathrm{Mod}(\xi,\Pi,E)$ has a left $\mathbb{V}$-adjoint $\mathrm{Mod}(f,\sigma,\pi)^{\$}$. This left $\mathbb{V}$-adjoint has the property that for any (ξ,Π,E)-model m (denoting $\mathrm{Mod}(f,\sigma,\pi)^{\$}(m)$ by $\pi^{\$}(m)$)*

$$\begin{tikzcd}
\{\omega_0, \gamma_0^{\spadesuit}(\pi^{\$}(m)^-)\} \arrow[r, "\cong"] \arrow[d, "\kappa_{\pi^{\$}(m)}"'] & \int_{n\models E'}[\mathrm{Mod}(\xi,\Pi,E)(m,n\pi), \{\omega_0, \gamma_0^{\spadesuit}(n^-)\}] \arrow[d, "{\int_{n\models E'}[1,\kappa_n]}"] \\
\{\omega_1, \gamma_1^{\spadesuit}(\pi^{\$}(m)^-)\} \arrow[r, "\cong"'] & \int_{n\models E'}[\mathrm{Mod}(\xi,\Pi,E)(m,n\pi), \{\omega_1, \gamma_1^{\spadesuit}(n^-)\}]
\end{tikzcd}$$

and

$$\begin{tikzcd} \{\omega_0, \gamma_0^{\spadesuit}(\pi^{\$}(m)^-)\} \arrow[r, "\cong"] \arrow[d, "p_{\pi^{\$}(m)^-}"'] & \int_{n \models E'} [\mathrm{Mod}(\xi, \Pi, E)(m, n\pi), \{\omega_0, \gamma_0^{\spadesuit}(n^-)\}] \arrow[d, "{n \models E'}[1, p_{n^-}]"] \\ Y(y, \pi^{\$}(m)^-\zeta) \arrow[r, "\cong"'] & \int_{n \models E'} [\mathrm{Mod}(\xi, \Pi, E)(m, n\pi), Y(y, n^-\zeta)] \end{tikzcd}$$

commute for any (ζ, Π')*-operation* $\kappa : \ (\omega_0, \gamma_0) \rightarrow (\omega_1, \gamma_1)$*, and any projection* $p : \ \{\omega_0, \gamma_0^{\spadesuit}\} \rightarrow Y(y, \zeta)$ *(with a similar square for any projection* $q : \ \{\omega_1, \gamma_1^{\spadesuit}\} \rightarrow Y(y, \zeta)$*).*

Corollary 5.14 gives conditions under which a theory has free models over variables.

Corollary 5.14 *Given theory* (M, ξ, X, Π, E)*, if* ξ *is right* $\mathbb{V}$*-adjoint and* M *is* $\mathbb{V}$*-complete,* $U_{\xi,\Pi,E} : \ \mathrm{Mod}(\xi, \Pi, E) \rightarrow X$ *has a left* $\mathbb{V}$*-adjoint* $U^{\$}_{\xi,\Pi,E}$*.*

5.6 Deduction

Definition 5.15 *A sentence of the form* $\forall x\, \kappa q = 1$ *or* $\exists \kappa \forall x\, \kappa q = 1$ *is called unconditional. Using the notation of Definition 5.8 note in particular this means* $\{\omega_0, \gamma_0^{\spadesuit}\} = X(x, \xi)$*.*

Definition 5.16 *Given a theory presentation* (ξ, Π, E) *containing only structural sentences, and an unconditional* (ξ, Π)*-sentence* $\forall x\, \kappa q = 1$*,* $\forall x\, \kappa q = 1$ *is deducible from* E *– written* $E \vdash \forall x\, \kappa q = 1$ *– if* $U^{\$}_{\xi,\Pi,E}(x) \models_{\xi,\Pi} \forall x\, \kappa q = 1$*. Similarly* $E \vdash \exists \kappa \forall x\, \kappa q = 1$ *if* $U^{\$}_{\xi,\Pi,E}(x) \models_{\xi,\Pi} \exists \kappa \forall x\, \kappa q = 1$

The notion of deduction presented in Definition 5.16 is motivated by the empirical fact that deduction of an equation $\forall x\, \phi$ from a theory E seems to take place in the free E-model (see [2] for example). The existence of such models is guaranteed by Corollary 5.14 (really a consequence of liberality), whilst the soundness of deduction of unconditional sentences based on truth in those models is proved in Theorem 5.17. The value of this result is that it reduces the problem of finding a sound and complete set of inference rules to that of finding a suitably inferential description of a *given* model.

Theorem 5.17 *Let* ξ *be right* $\mathbb{V}$*-adjoint,* E *be a* (ξ, Π)*-theory presentation containing only structural sentences, and* ϕ *be an unconditional* (ξ, Π)*-sentence. Then* $E \models_{\xi,\Pi} \phi$ *iff* $E \vdash_{\xi,\Pi} \phi$*.*

Proof. Suppose $E \models_{\xi,\Pi} \phi$. By Corollary 5.14, $U^{\$}_{\xi,\Pi,E}(x) \models E$, whence $U^{\$}_{\xi,\Pi,E}(x) \models \phi$, i.e., $E \vdash_{\xi,\Pi} \phi$. For the converse suppose first that ϕ is existential, i.e., suppose $U^{\$}_{\xi,\Pi,E}(x) \models \exists \kappa \forall x\, \kappa q = 1$, and let m be a (ξ, Π)-model satisfying E. With reference to

$$\begin{array}{ccc}
\int_{m\models E}[X(x,m^-\xi),X(x,m^-\xi)] & \xrightarrow{m\iota} & [X(x,m^-\xi),X(x,m^-\xi)] \\
{\scriptstyle f}\downarrow\ \uparrow{\scriptstyle \int_{m\models E}[1,q_{m^-}]} & (*) & {\scriptstyle [1,f_m]}\downarrow\ \uparrow{\scriptstyle [1,q_{m^-}]} \\
\int_{m\models E}[X(x,m^-\xi),\{\omega,\gamma_1^{\spadesuit}(m^-)\}] & \xrightarrow[m\iota]{} & [X(x,m^-\xi),\{\omega,\gamma_1^{\spadesuit}(m^-)\}]
\end{array}$$

and by Theorem 5.13 and Corollary 5.14 this means there exists an f such that $f;\int_{m\models E}[1,q_{m^-}]=1$. Hence

$$m\iota = f;\int_{m\models E}[1,q_{m^-}];m\iota = f;m\iota;[1,q_{m^-}] \tag{1}$$

Let f_m correspond to $u;f;m\iota$ by adjunction, i.e., define f_m by the equation $u;f;m\iota = u_m;[1,f_m]$, where $u:\ e\to\int_{m\models E}[X(x,m^-\xi),X(x,m^-\xi)]$ is the identity $\mathbb{V}$-natural transformation $X(x,U_{\xi,\Pi,E})\to X(x,U_{\xi,\Pi,E})$. Then $u_m = u;m\iota = u;f;m\iota;[1,q_{m^-}] = u_m;[1,f_m;q_{m^-}]$, i.e., $f_m;q_{m^-}=1$. Therefore $m\models \exists\kappa\,\forall x\,\kappa q=1$.

Now suppose ϕ is structural, so let ϕ be $\forall x\,\kappa q=1$. In this case the f in $(*)$ above is $\int_{m\models E}[1,\kappa_m]$, and hence by (1)

$$m\iota = \int_{m\models E}[1,\kappa_m];m\iota;[1,p_{m^-}] = m\iota;[1,\kappa_m];[1,p_{m^-}]$$

which yields $\kappa_m;p_{m^-}=1$ as before, establishing $m\models\forall x\,\kappa q=1$. □

5.7 Exactness

This section presents two important "preservation" theorems: Theorem 5.20 states that "adding" USX operations to a cocomplete category of signatures results in a cocomplete category; Theorem 5.21 states that a models functor cocontinuous on a category of signatures is still cocontinuous when extended to signatures enriched with USX operations. To bring these results into play Definition 5.18 formalises the idea that models indexed by signatures have underlying domains, and Proposition 5.19 shows how the "addition" of USX operations to given signatures canonically determines an institution.

Definition 5.18 *A models* $\mathbb{V}$*-functor* $\mathrm{Mod}:\ \mathrm{Sign}\to\mathbb{V}\mathrm{Cat}^{\mathrm{op}}$ *has underlying domains if there exists a functor* $U:\ \mathrm{Sign}\to\mathbb{V}\mathfrak{Sig}$ *such that* $\mathrm{Mod}=U;\partial_1$*, where* ∂_1 *maps signatorials* $\xi:\ M\to X$ *to* M *and signatorial morphisms* $(f,\sigma):\ (M,\xi,X)\to(N,\zeta,Y)$ *to* $\sigma:\ N\to M$*.*

Proposition 5.19 *A models* $\mathbb{V}$*-functor* $\mathrm{Mod}:\ \mathrm{Sign}\to\mathbb{V}\mathrm{Cat}^{\mathrm{op}}$ *with underlying domains* U *determines a USX fragment USX*(U)*.*

Proof. Let $\mathrm{Dom}:\ \mathrm{Sign}\to\mathbb{V}\mathrm{Cat}^{\mathrm{op}}$ be the domains $\mathbb{V}$-functor, and let the models-to-domains $\mathbb{V}$-functors (collectively a natural transformation) be given by $U : \ \mathrm{Mod} \to \mathrm{Dom}$. Then define $\mathrm{Sign}_{\mathsf{USX}(U)}$ to be the category whose objects are pairs (Σ, Π) such that $\Sigma \in |\,\mathrm{Sign}\,|$ and $(U_\Sigma, \Pi) \in |\,\mathrm{Sign}_{\mathsf{USX}}\,|$, and whose arrows $(\Sigma, \Pi) \to (\Sigma', \Pi')$ are pairs (σ, π) such that $\sigma : \ \Sigma \to \Sigma' \in \mathrm{Sign}$ and $(U_\sigma, \pi) : \ (U_\Sigma, \Pi) \to (U_{\Sigma'}, \Pi') \in \mathrm{Sign}_{\mathsf{USX}}$. Composition is pointwise: $(\sigma, \pi); (\sigma', \pi') = (\sigma\sigma', \pi\pi')$.

Next define $\Phi_{\mathsf{USX}(U)} : \ \mathrm{Sign}_{\mathsf{USX}(U)} \to \mathrm{Sign}_{\mathsf{USX}}$ by the mappings $(\Sigma, \Pi) \mapsto (U_\Sigma, \Pi)$ and $(\sigma, \pi) \mapsto (U_\sigma, \pi)$. Finally set $\mathrm{Mod}_{\mathsf{USX}(U)} : \ \mathrm{Sign}_{\mathsf{USX}(U)} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ equal to the composite $\Phi_{\mathsf{USX}(U)}; \mathrm{Mod}_{\mathsf{USX}}$. $\square$

Theorem 5.20 *Let* $\mathrm{Mod} : \ \mathrm{Sign} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ *have underlying domains* U*. Then* $\mathrm{Sign}_{\mathsf{USX}(U)}$ *is cocomplete if* Sign *is cocomplete.*

Theorem 5.21 *Let* $\mathrm{Mod} : \ \mathrm{Sign} \to \mathbb{V}\mathbb{C}\mathrm{at}^{\mathrm{op}}$ *be a models functor with underlying domains* U*. Then* $\mathsf{USX}(U)$ *is exact if* Mod *is cocontinuous.*

6 Rewriting Logic in a 2-Category

In this section $\mathbb{V} = \mathbb{C}\mathrm{at}$ – the universe appropriate for rewriting logic as the remainder of this paper attempts to demonstrate. The idea is to identify a generic rewriting institution 2-RWL as a fragment of $\mathbb{C}$at-USX, and then to show that the former does in fact include traditional rewriting logic as a special case.

6.1 2-RWL Signatures

The first step is to identify the 2-RWL signatures amongst the $\mathbb{C}$at-USX signatures. In fact they will be characterised as those $\mathbb{C}$at-USX signatures (ξ, Π) satisfying

$$\Pi_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)} = \emptyset \text{ whenever } \neg\Phi(\omega_0, \gamma_0, \omega_1, \gamma_1) \qquad (\dagger)$$

for some property Φ of ω_0, γ_0, ω_1 and γ_1.

Definition 6.1 *For* $k \geq 1$ *define*

- Γ^k *to be the 2-category generated by the graph*

$$\begin{tikzcd}
 & o_1 \\
o_0 \arrow[ur, shift left, "v_1"] \arrow[ur, shift right, "w_1"'] \arrow[dr, shift left, "v_k"] \arrow[dr, shift right, "w_k"'] & \vdots \\
 & o_k
\end{tikzcd}$$

- *2-functor* $\omega^k : \ \Gamma^k \to \mathbb{C}\mathrm{at}$ *by the mapping*
  - $\cdot$ $o_0 \mapsto 1$
  - $\cdot$ $o_i \mapsto 2$ *for* $i = 1, \cdots, k$
  - $\cdot$ $v_i \mapsto (0 \mapsto 0) : \ 1 \to 2$ *for* $i = 1, \cdots, k$
  - $\cdot$ $w_i \mapsto (0 \mapsto 1) : \ 1 \to 2$ *for* $i = 1, \cdots, k$

where 2 *is the 2-category generated by the graph* $0 \to 1$.

To help make sense of Definition 6.2 below recall the conceptualisation in the Introduction of the semantics of a conditional equation as a function between two limits (the sets of solutions of the antecedent and consequent). Clause (i) implies that the carriers of the models are objects in an arbitrary 2-category, not restricted to being (sorted) categories. Furthermore it says that the underlying equational logic (in the sense of models, terms and their denotations) is left *unspecified*. Clauses (ii) (see Definition 6.1) and (iii) taken together imply that the diagrams of substitutions are those corresponding to equations and rewrite rules, both conditional ($\Gamma_0 = \Gamma^k$) and unconditional ($\Gamma_0 = 1$). The very different solution sets associated respectively with equations and rewrite rules arise upon choosing different weights: diagonal functors for equations yielding ordinary (see section 2.0.4) equalisers (simultaneous for $k > 1$); and the 2-functors ω^k yielding (simultaneous for $k > 1$) subequalisers (see [11], page 19). The various combinations are listed in clause (iv).

Definition 6.2 $\Phi(\omega_0, \gamma_0, \omega_1, \gamma_1)$ *is the conjunction of the following clauses:*

(i) ω_i*,* γ_i *are 2-functors* $\omega_i : \ \Gamma_i \to \mathbb{C}\mathrm{at}$*,* $\gamma_i : \ \Gamma_i \to \mathcal{K}(\xi)^{\mathrm{op}}$ *(for* $i = 0, 1$*) where* $\xi : \ M \to X$ *is an arbitrary 2-functor.*

(ii) $\Gamma_0 = \Gamma^k$ *(for some* $k \leq 1$*), or* 1 *(containing only one object* 0 *and no non-trivial arrows); and* $\Gamma_1 = \Gamma^1$*.*

(iii) γ_0 *and* γ_1 *are arbitrary subject to the constraint*

$$\gamma_1(o_0) = \begin{cases} \gamma_0(0) & \textit{if } \Gamma_0 = 1 \\ \gamma_0(o_0) & \textit{if } \Gamma_0 = \Gamma^k \end{cases}$$

(iv) *The pair* (ω_0, ω_1) *is one of the following combinations:*

(a) $(\Delta_1 : \ 1 \to \mathbb{C}\mathrm{at}, \Delta_1 : \ \Gamma^1 \to \mathbb{C}\mathrm{at})$ *(unconditional equations)*

(b) $(\Delta_1 : \ \Gamma^k \to \mathbb{C}\mathrm{at}, \Delta_1 : \ \Gamma^1 \to \mathbb{C}\mathrm{at})$ *(conditional equations)*

(c) $(\Delta_1 : \ 1 \to \mathbb{C}\mathrm{at}, \omega^1 : \ \Gamma^1 \to \mathbb{C}\mathrm{at})$ *(unconditional rewrite rules)*

(d) $(\omega^k : \ \Gamma^k \to \mathbb{C}\mathrm{at}, \omega^1 : \ \Gamma^1 \to \mathbb{C}\mathrm{at})$ *(conditional rewrite rules)*

Thus 2-RWL signatures contain operations only between domains and codomains which make sense for rewriting logic:

Definition 6.3 *Definition 6.2 and* (†) *determine a full subcategory* $\mathrm{Sign}_{\textsf{2-RWL}}$ *of* $\mathrm{Sign}_{\textsf{Cat}-\textsf{USX}}$.

Note that non-conventional rewriting logics might admit also the following weight combinations (for example) corresponding to the indicated sentences:

- $(\Delta_1 : \ \Gamma^k \to \mathbb{C}\mathrm{at}, \omega^1 : \ \Gamma^1 \to \mathbb{C}\mathrm{at})$ (*rewrite rules conditional on equations*)
- $(\omega^k : \ \Gamma^k \to \mathbb{C}\mathrm{at}, \Delta_1 : \ \Gamma^1 \to \mathbb{C}\mathrm{at})$ (*equations conditional on rewrite rules*)

similarly yielding a category $\mathrm{Sign}'_{\textsf{2-RWL}}$, of which $\mathrm{Sign}_{\textsf{2-RWL}}$ is a (full) subcategory.

6.2 The 2-RWL Institution

Consider a $\mathbb{C}$at-USX sentence $(\exists\kappa)\forall x\,\kappa q = p$ over a $\mathrm{Sign}_{\textsf{2-RWL}}$ signature (ξ,Π). With reference to Definitions 5.8 and 6.2 the pullback of γ_0 and γ_1 must be 1, with $u_0(0) = 0$ or o_0, depending on whether $\Gamma_0 = 1$ or Γ^k (Definition 6.1); and $u_1(0) = o_0$. By Definition 6.1 it follows that $0u_0\omega_0 = 1 = 0u_1\omega_1$, whence by Definition 5.8, clause (iii), the projections p and q *must* correspond to the unique functor $1 \to 1$ (remembering that $e = 1$), i.e., the identity functor on 1. Denoting these projections respectively by p_x and q_x prompts

Definition 6.4 *Define* $\mathrm{Sen}_{\textsf{2-RWL}}: \ \mathrm{Sign}_{\textsf{2-RWL}} \to \mathbb{S}\mathrm{et}$ *to be the restriction of* $\mathrm{Sen}_{\mathbb{C}\mathrm{at}-\textsf{USX}}$ *to* $\mathrm{Sign}_{\textsf{2-RWL}}$*. Similarly define* $\mathrm{Sen}'_{\textsf{2-RWL}}$ *be the restriction of* $\mathrm{Sen}_{\mathbb{C}\mathrm{at}-\textsf{USX}}$ *to* $\mathrm{Sign}'_{\textsf{2-RWL}}$*.*

which, together with Definition 6.3 and Proposition 4.4, yields

Corollary 6.5 *2-RWL and 2-RWL′ are* $\mathbb{C}$at*-institutions.*

6.3 Identification of 2-RWL Syntax

At this stage it is instructive to identify 2-RWL in more familiar terms, beginning with the sentences. Firstly note that there is a bijection between the set of *structural* (ξ,Π) sentences and the set of all (ξ,Π)-operations (by the argument leading up to Definition 6.4); secondly for any two $\forall x\,\kappa q_x = p_x$, $\forall x\,\kappa' q_x = p_x$ with $\kappa,\kappa': \ (\omega_0,\gamma_0) \to (\omega_1,\gamma_1)$ such that (ω_0,ω_1) corresponds iv.(a) or iv.(b) in Definition 6.2, if $m \models \forall x\,\kappa q_x = p_x$ and $m \models \forall x\,\kappa' q_x = p_x$ then $\kappa_m = \kappa'_m$ (because the codomain is an equaliser); and thirdly that the substitutions diagrams $\gamma_0: \ \Gamma^k \to \mathcal{K}(\xi)^{\mathrm{op}}$ and $\gamma_1: \ \Gamma^1 \to \mathcal{K}(\xi)^{\mathrm{op}}$ are wholly determined by substitutions $t_i, t'_i: \ y_i \to xT_\xi$ and $t, t': \ y \to xT_\xi$ subject to the equations $x = \gamma_0(o_0)$, $y_i = \gamma_0(o_i)$, $t_i = \gamma_0(v_i)$, $t'_i = \gamma_0(w_i)$, $y = \gamma_1(o_1)$, $t = \gamma_1(v_1)$, and $t' = \gamma_1(w_1)$, for $i = 1 \cdots k$. This motivates

Definition 6.6 *The category 2-RWL-*Spec *of rewriting specifications has as objects specifications* (ξ,Π,E) *in* $\mathrm{Spec}_{\textsf{2-RWL}}$ *(see Definition 4.2) such that* κ *equals the formal string* eq *if* κ *corresponds to (iv).a or (iv).b of Definition 6.2, and* E *equals the set* $\{\forall x\,\kappa q_x = p_x | \kappa \in \Pi\}$*. As arrows it has morphisms* $(f,\sigma,\pi): \ (\xi,\Pi,E) \to (\zeta,\Pi',E')$ *in* $\mathrm{Spec}_{\textsf{2-RWL}}$ *such that* $\pi(\mathrm{eq}) = \mathrm{eq}$*. 2-RWL′-*Spec *may be defined similarly.*

Rewriting sentences may thus be denoted more suggestively as follows (using the notation of the previous paragraph):

$$\begin{array}{rcl}
\forall x\, t = t' & \sim & \forall x\ \mathrm{eq}: \ (\Delta_1,\gamma_0) \to (\Delta_1,\gamma_1)\,;\ q_x = p_x \\
\forall x\, t = t' \,\mathrm{if}\, t_i = t'_i & \sim & \forall x\ \mathrm{eq}: \ (\Delta_1,\gamma_0) \to (\Delta_1,\gamma_1)\,;\ q_x = p_x \\
r: \ \forall x\, t \to t' & \sim & \forall x\, r: \ (\Delta_1,\gamma_0) \to (\omega^1,\gamma_1)\,;\ q_x = p_x \\
r: \ \forall x\, t \to t' \,\mathrm{if}\, t_i \to t'_i & \sim & \forall x\, r: \ (\omega^k,\gamma_0) \to (\omega^1,\gamma_1)\,;\ q_x = p_x
\end{array}$$

including the two non-conventional ones:

$$\begin{array}{lcl} r:\ \forall x\, t \to t' \,\text{if}\, t_i = t_i' & \sim & \forall x\, r:\ (\Delta_1, \gamma_0) \to (\omega^1, \gamma_1)\,;\ q_x = p_x \\ r:\ \forall x\, t = t' \,\text{if}\, t_i \to t_i' & \sim & \forall x\, r:\ (\omega^k, \gamma_0) \to (\Delta_1, \gamma_1)\,;\ q_x = p_x \end{array}$$

It also follows from Definition 6.6 that rewriting specifications (ξ, Π, E) may be represented in the traditional manner as a triple (ξ, E', R) by separating the equations from the rewrite rules; conversely a traditional triple (ξ, E, R) is equivalent to a triple $(\xi, \mathrm{op}(E, R), \mathrm{sen}(E, R))$ defined in the obvious way.

Finally note that Definition 6.6 only deals with structural sentences (sentences of the form $\forall x\, \kappa q_x = p_x$); existential ones (of the form $\exists \kappa\, \forall x\, \kappa q_x = p_x$) are redundant in the case of (conditional) equations as they amount to the same thing as structural ones (see the paragraph following Lemma 6.8). Existential conditional rewrite rules do not appear in [11]; unconditional ones do, and are called *sequents.*

6.4 Identification of 2-RWL Semantics

Lemma 6.7 *Given a family of functors $F_i, G_i:\ A \to B_i$ for $i = 1, \cdots, k$ the simultaneous subequaliser of this family (page 19 in [11]) is the weighted $\mathbb{C}\mathrm{at}$-limit (2-limit) $\{\omega^k, H\}$ where $H:\ \Gamma^k \to \mathbb{C}\mathrm{at}$ is defined by the equations $H(o_0) = A, H(o_i) = B_i, H(v_i) = F_i, H(w_i) = G_i$ for $i = 1, \cdots, k$. The subequaliser functor [11] is the projection corresponding to the identity functor $1 \to 1$.*

Proof. Straightforward identification of $\{\omega^k, H\}$. □

Lemma 6.8 *Given (ω^k, γ_0) and (ω^1, γ_1) such that $\Phi(\omega^k, \gamma_0, \omega^1, \gamma_1)$ (see Definition 6.2, (iv)) there is a 1-1 correspondence between natural transformations $\theta:\ p_x(m^-); t(m) \to p_x(m^-); t'(m)$, and functors $f:\ \{\omega^k, \gamma_0^{\spadesuit}(m^-)\} \to \{\omega^1, \gamma_1^{\spadesuit}(m^-)\}$ satisfying $f; q_x(m^-) = p_x(m^-)$.*

Proof. With reference to

$$\begin{tikzcd} & \{\omega^k, \gamma_0^{\spadesuit}(m^-)\} \arrow[d, "p_x(m^-)"] \arrow[dr, "f"] & \\ X(y, m\xi) & X(x, m\xi) \arrow[l, shift right, "t'(m)"'] \arrow[l, shift left, "t(m)"] & \{\omega^1, \gamma_1^{\spadesuit}(m^-)\} \arrow[l, "q_x(m^-)"] \end{tikzcd}$$

let $\psi:\ q_x(m^-); t(m) \dot{\to} q_x(m^-); t'(m)$ be the universal natural transformation associated with $\{\omega^1, \gamma_1^{\spadesuit}(m^-)\}$. Then the 1-1 correspondence $\theta \leftrightarrow f$ where $\theta:\ p_x(m^-); t(m) \to p_x(m^-); t'(m)$ and f satisfies $f; q_x(m^-) = p_x(m^-)$ follows immediately from the universality of ψ . □

Analogous results to Lemma 6.8 are obtained for the other cases: for the combination $(\Delta_1:\ 1 \to \mathbb{C}\mathrm{at}, \gamma_0)$ and (ω^1, γ_1), $p_x(m^-) = 1$ and $f \leftrightarrow$

$\theta : t(m) \to t'(m)$; for $(\Delta_1 : \Gamma^k \to \mathbb{C}\mathrm{at}, \gamma_0)$ and (Δ_1, γ_1), $f; q_x(m^-) = p_x(m^-)$, and $\therefore f$ is an inclusion of categories; for $(\Delta_1 : 1 \to \mathbb{C}\mathrm{at}, \gamma_0)$, (Δ_1, γ_1), $p_x(m^-) = 1$, $\therefore f; q_x(m^-) = 1$, $\therefore f = 1$ by universality. As regards the non-conventional sentences, $(\Delta_1 : \Gamma^k \to \mathbb{C}\mathrm{at}, \gamma_0)$ and (ω^1, γ_1) means $p_x(m^-)$ is an inclusion, and $\therefore f \leftrightarrow \theta : p_x(m^-); t(m) \to p_x(m^-); t'(m)$; finally for (ω^k, γ_0) and (Δ_1, γ_1), $q_x(m^-)$ is an inclusion, and f has following property: if $v : x \to m\xi$ such that there exist $t_i(m)(v) \to t_i'(m)(v)$ then $t(m)(f(v)) = t'(m)(f(v))$.

Proposition 6.9 below confirms that the specifications of Definition 6.6 determine precisely the R-systems, R-system morphisms and R-system modifications (when these are generalised in the obvious way) of [11].

Proposition 6.9 *(Identification of RWL rewrite systems) Given signatorial $\xi : M \to X$, a set E of ξ-equations, and a set R of rewrite rules, there exists an isomorphism*

$$\mathrm{Alg}(\xi, E, R) \cong \mathrm{Mod}(\xi, \mathrm{op}(E, R), \mathrm{sen}(E, R))$$

Proof. Given a (ξ, E, R) system m construct a $\mathrm{Mod}(\xi, \mathrm{op}(E, R))$-model m_+ by letting the interpretation of every operation in $\mathrm{op}(E, R)$ correspond, under the bijection of Lemma 6.8, to the interpretation in m of the corresponding equation in E or rewrite rule in R. For example let the interpretation in m_+ of the conditional rewrite rule $r : \forall x\, t \to t' \,\mathrm{if}\, t_i \to t_i'$, considered as an operation in $\mathrm{op}(E, R)$, correspond to its interpretation in m, which is a natural transformation $\theta : p_x(m^-); t(m) \to p_x(m^-); t'(m)$. By construction these interpretations in m_+ satisfy the equations in $\mathrm{sen}(E, R)$. Also by construction m and m_+ have the same underlying ξ-model m^-. This sets up a bijection between the (ξ, E, R)-systems and $(\xi, \mathrm{op}(E, R), \mathrm{sen}(E, R))$-models.

Let m and n be (ξ, E, R)-systems. The objective is to show

$$\mathrm{Alg}(\xi, E, R)(m, n) \cong \mathrm{Mod}(\xi, \mathrm{op}(E, R), \mathrm{sen}(E, R))(m_+, n_+) \tag{2}$$

in $\mathbb{C}\mathrm{at}$. Homomorphisms and modifications (natural transformations between the homomorphisms) will be treated separately. From the definition of $\mathrm{Mod}(\xi, \mathrm{op}(E, R), \mathrm{sen}(E, R))(m_+, n_+)$ (Definition 5.4), homomorphisms $h : m_+ \to n_+$ are precisely those ξ-homomorphisms $h : m^- \to n^-$ such that the square

$$\begin{tikzcd}
\{\omega_0, \gamma_0^{\spadesuit}(m^-)\} \arrow[r, "\{\omega_0, \gamma_0^{\spadesuit}(h)\}"] \arrow[d, "\phi_{m_+}"'] \arrow[dr, phantom, "(*)"] & \{\omega_0, \gamma_0^{\spadesuit}(n^-)\} \arrow[d, "\phi_{n_+}"] \\
\{\omega_1, \gamma_1^{\spadesuit}(m^-)\} \arrow[r, "\{\omega_1, \gamma_1^{\spadesuit}(h)\}"'] & \{\omega_1, \gamma_1^{\spadesuit}(n^-)\}
\end{tikzcd}$$

commutes for each $\phi : (\omega_0, \gamma_0) \to (\omega_1, \gamma_1) \in \mathrm{op}(E, R)$. It suffices to show that these correspond exactly to those homomorphisms $h : m^- \to n^-$ which "preserve R" ([11], page 22).

Suppose $r: \forall x\, t \to t'$ if $t_i \to t'_i$ is a conditional rewrite rule in R. Denote by ψ^k_m, ψ^k_n, ψ^1_m, ψ^1_n the universal natural transformations associated respectively with $\{\omega^k, (\gamma_0)^{\spadesuit}(m^-)\}$, $\{\omega^k, (\gamma_0)^{\spadesuit}(n^-)\}$, $\{\omega^1, (\gamma_1)^{\spadesuit}(m^-)\}$ and $\{\omega^1, (\gamma_1)^{\spadesuit}(n^-)\}$. For $h: m^- \to n^-$ to preserve r means $r_m; X(1,h) = h^\bullet; r_n$ where $h^\bullet$ is the unique functor induced by the universal property of $\{\omega^k, (\gamma_0)^{\spadesuit}(n^-)\}$ and the natural transformation $\psi^k_n \,⨟\, X(y,h)$ ([11], page 22); in fact $h^\bullet$ is the functor $\{\omega^k, (\gamma_0)^{\spadesuit}(h)\}: \{\omega^k, (\gamma_0)^{\spadesuit}(m^-)\} \to \{\omega^k, (\gamma_0)^{\spadesuit}(m^-)\}$. So h preserves r is equivalent to the equation

$$r_m; X(1,h) = \{\omega^k, (\gamma_0)^{\spadesuit}(h)\}; r_n \tag{3}$$

Now

$$\begin{aligned} r_m X(1,h) &= r_{m_+} \psi^1_m X(1,h) && \text{(Lemma 6.8)} \\ &= r_{m_+} \{\omega^1, (\gamma_1)^{\spadesuit}(h)\} \psi^1_n && \text{(Property of weighted limit)} \end{aligned}$$

whilst $\{\omega^k, (\gamma_0)^{\spadesuit}(h)\} r_n = \{\omega^k, (\gamma_0)^{\spadesuit}(h)\} r_{n_+} \psi^1_n$ by Lemma 6.8. It follows from the universality of ψ^1_n that (3) iff $(*)$ with $\phi = r$. The case of unconditional rewrite rules is similar, whereas in the case of equations $(*)$ is automatically satisfied.

As regards the modifications between m and n, these are all the modifications between m^- and n^- ([11], page 22). The fact that *all* modifications between m^- and n^- satisfy $(*)$ (a property of the weighted limit) therefore establishes (2). □

6.5 Rewriting Logic in $\mathbb{C}$at

The institution of rewriting logic in $\mathbb{C}$at is obtained by another application of Proposition 4.4. $\mathrm{Sign}_{\mathsf{RWL}}$ is the full subcategory of $\mathrm{Sign}_{\mathsf{2\text{-}RWL}}$ containing only those 2-RWL signatures (ξ, Π) such that

(i) ξ is of the form $M \to [S, \mathbb{C}\mathrm{at}]$ where S is discrete;

(ii) $\Pi_{(\omega_0,\gamma_0),(\omega_1,\gamma_1)} = \emptyset$ *unless* (a) $\gamma_0(o_0) = S(s_i, \_)$ for $i = 1 \cdots k$ and for some s_i in S, whenever $\Gamma_0 = \Gamma^k$; (b) $\gamma_1(o_1) = S(s, \_)$ for some s in S; and (c) x is discrete.

Clause (i) says that the carriers of the models are sorted categories, but still leaves unspecified the models themselves; clauses (ii.a) and (ii.b) say that the substitutions used to build sentences correspond to single terms; and clause (ii.c) says that the variables over which equations and rewrite rules are quantified form a sorted *set*, i.e., there are no arrow variables.

The sentence functor $\mathrm{Sen}_{\mathsf{RWL}}$ is defined to be the restriction of $\mathrm{Sen}_{\mathsf{2\text{-}RWL}}$ to $\mathrm{Sign}_{\mathsf{RWL}}$, i.e., over RWL signatures, RWL sentences are the same as 2-RWL ones. The resulting institution is denoted by RWL.

By the 2-Yoneda Lemma $[\![S(s,\_), xT_\xi]\!] \cong T_\xi(x)_s$, rendering RWL sentences in an even more familiar form: for example the conditional rewrite rule becomes $r: \ \forall x\, t \rightarrow t' \,\mathrm{if}\, t_i \rightarrow t'_i$ where $t, t' \in |T_\xi(x)_s|$, and $t_i, t'_i \in |T_\xi(x)_{s_i}|$ for some $s, s_i \in S$, $i = 1 \cdots k$. Its denotation in an R-system is then a natural transformation $\theta: \ p_x(m^-); t(m) \rightarrow p_x(m^-); t'(m)$ where $t(m)$ and $t'(m)$ are of the form $[\![x, m\xi]\!] \rightarrow m_s$ (because $[\![S(s, \_), m\xi]\!] \cong \xi(m)_s$).

7 Conclusions

This paper unifies in a single abstract framework, an institution called USX, the formal semantics of both equational and rewriting logic, revealing that the syntax and semantics extends without much effort to domains which lie in an arbitrary 2-category rather than $\mathbb{C}$at, resulting in an institution 2-RWL of "rewriting logic in a 2-category". The identification of 2-RWL as an *institution fragment* – itself possibly a new notion – of USX implies that properties enjoyed by USX are automatically conferred upon 2-RWL. These include liberality and the existence of free models over variables; exactness; and a kind of complete deduction result: to find a sound and complete set of inference rules it is sufficient to find inference rules for deciding truth in a certain model which is guaranteed to exist.

Furthermore the uniform syntax and semantics provided by USX allows the seamless addition of new kinds of sentences to 2-RWL without altering its status as a USX fragment. It also renders 2-RWL independent of the underlying equational logic.

7.1 Future Work

Membership Logic has not been considered at all in this paper, but in view of its close ties with rewriting logic, it ought to be at some stage.

7.2 Omitted Literature

I apologize for not having discussed the work in [10] in relation to 2-categorical notions of rewriting; and also in relation to theory morphisms. The reason is that I never read [11] carefully enough to spot the reference, until several days before the submission of this paper, by which time it was too late to acquire a copy. Once again I apologize, to the reader, and to José Meseguer.

It has also been pointed out to me that Hiroyuki Miyoshi at the Computing Science Department at Kyoto Sangyo University in Kyoto, Japan, has given a 2-categorical semantics for rewriting logic [12]. My sincere apologies to Hiroyuki Miyoshi also.

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